The Cauchy problem of the Ward equation

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Abstract:

Taking a dimension reduction and a gauge fixing of the self-dual Yang-Mills equation in the space-time with signature $(2,2)$, one derives an integrable system, the Ward equation:

$$-\left(J^{-1}J_t\right)_t+\left(J^{-1}J_x\right)_x+\left(J^{-1}J_y\right)_y+\left\{\left(J^{-1}J_t\right)_y-\left(J^{-1}J_y\right)_t\right\}=0, \quad J(x,y,t)\in SU(N).$$

Our main contribution is solving the inverse scattering problem and the Cauchy problem of the Ward equation without small data constraints. More precisely, we generalize the results in [Dai-Terng, 07], [Villarroel, 90], [Fokas-Ioannidou, 01], [Dai-Terng-Uhlenbeck, 06] by showing:

1. Suppose the potential Q is in P. Then there is a bounded set $Z\subset \mathbf{C}$, where $Z\cap(\mathbf{C}\backslash\mathbf{R})$ is discrete in $\mathbf{C}\backslash\mathbf{R}$, such that the associated eigenfunction $\psi(x,y,t,\lambda)$ exists uniquely for $\lambda\in\mathbf{C}\backslash\left(\mathbf{R}\cup Z\right)$ and is λ-meromorphic with poles at $Z\cap(\mathbf{C}\backslash\mathbf{R})$.
2. A notion of scattering data v is formulated for $Q\in P$ if Z is empty. The algebraic and analytic characterization of such scattering data v is derived. Conversely, for each data v satisfying these constraints, there is a corresponding potential Q satisfying certain decay at ∞ and smoothness.
3. Suppose the initial potential Q_0 is in P, Z_0, the set of poles of $\psi(x,y,0,\lambda)$, is of finite number and $Z\subset(\mathbf{C}\backslash\mathbf{R})$. Then the Cauchy problem of the Ward equation admits a global solution $Q(x,y,t)$ satisfying certain decay at ∞ and smoothness.

A one-to-one correspondence between Q and J, a precise definition of the potential space P, and a description of the smoothness and decay at ∞ are provided in [Wu, 08], [Wu, 09]. Moreover, important algebraic properties of the Lax pair, (i) derivation property; (ii) translating invariant property; (iii) the principal part being equivalent to a $\overline{\partial}$-operator, are used in resolving the large data difficulty. Hence the Result 1, and the existence of Q in 2 are reduced to solving two types of Riemann-Hilbert problem with large data.
If the set of poles of $\psi(x,y,0,\lambda)$ is of finite number and contained in $\mathbf{C}\backslash\mathbf{R}$, then we use Result 2 to decompose $\psi(x,y,0,\lambda)$ as two factorization: one factorization has a rational factor g_0 as its tail, and the other has a holomorphic (in $\mathbf{C}\backslash\mathbf{R}$) tail f_0. Using f_0 and g_0 as initial eigenfunctions, we then solve the Cauchy problems of the Ward equation by Result 2 and the theory of Backlund transformation established in [Dai-Terng,07]. From these two solutions, we can construct the solution of Result 3 by using Backlund transformation again.